A Fast Explicit Operator Splitting Method for Modified Buckley-Leverett Equations

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Outlines

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Oil Recovery

In fluid dynamics, the Buckley-Leverett (BL) equation [1] is a simple model for two-phase flow in porous medium.

One application is secondary recovery by water-drive in oil reservoir simulation.

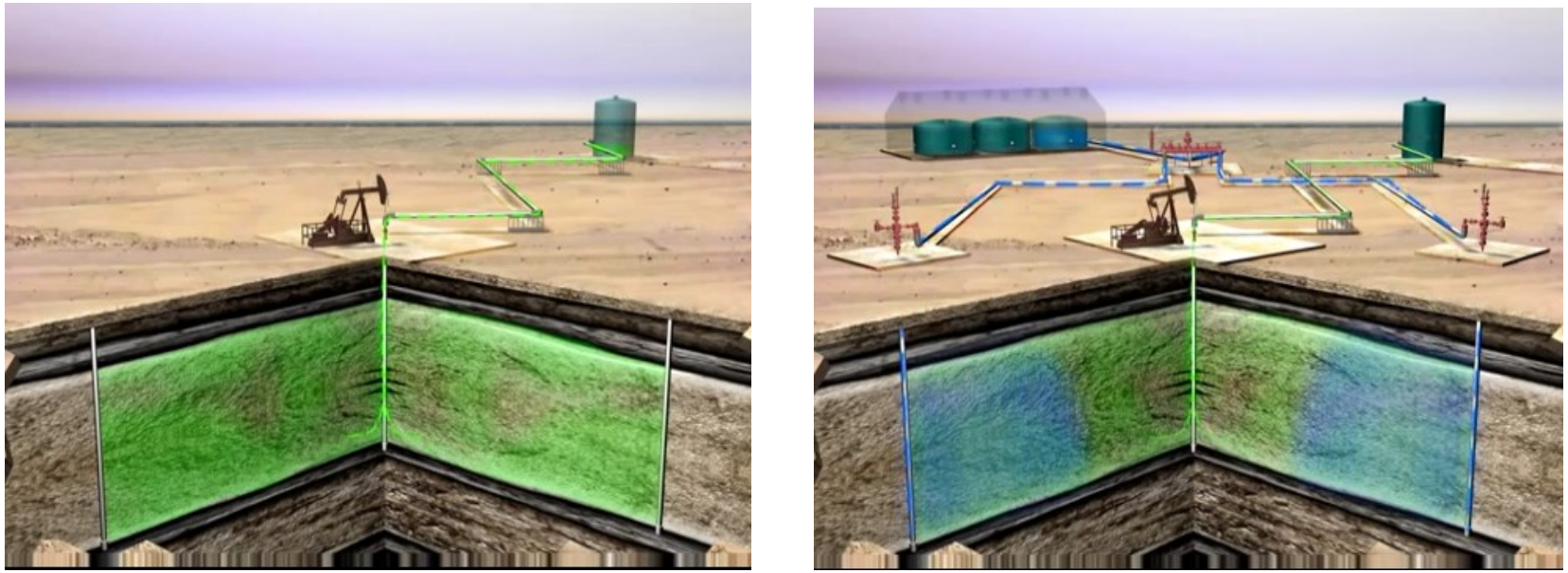


Figure: Primary recovery stage (5-15%) and secondary recovery stage (35-45%).

BL Equation

In the one-dimensional (1-D) case, the classical BL equation is a scalar conversation law

$$u_t + f(u)_x = 0,$$

with the flux function $f(u)$ being defined as

$$f(u) = \begin{cases} 0, & u < 0, \\ \dfrac{u^2}{u^2 + M(1-u)^2}, & 0 \le u \le 1, \\ 1, & u > 1. \end{cases}$$

u denotes the water saturation $0 \le u \le 1$. $u = 0$ pure oil. $u = 1$ pure water.
$M > 0$ the viscosity ratio between water and oil.

Flux

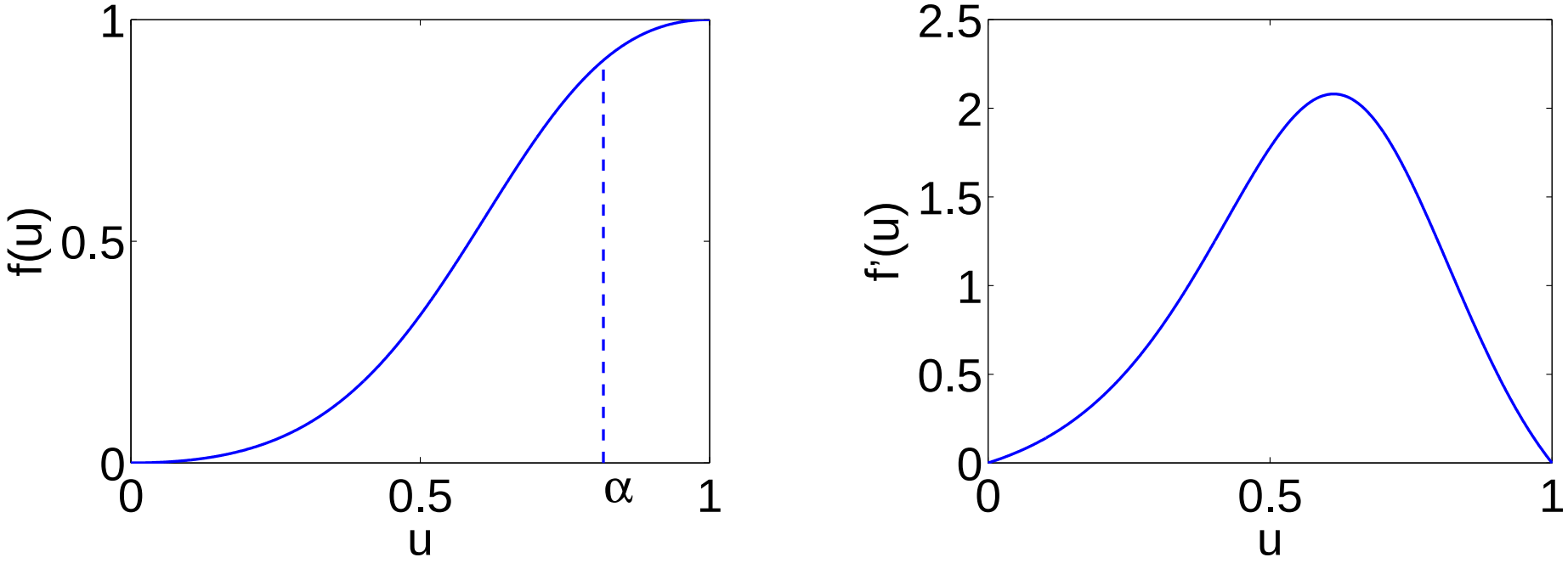


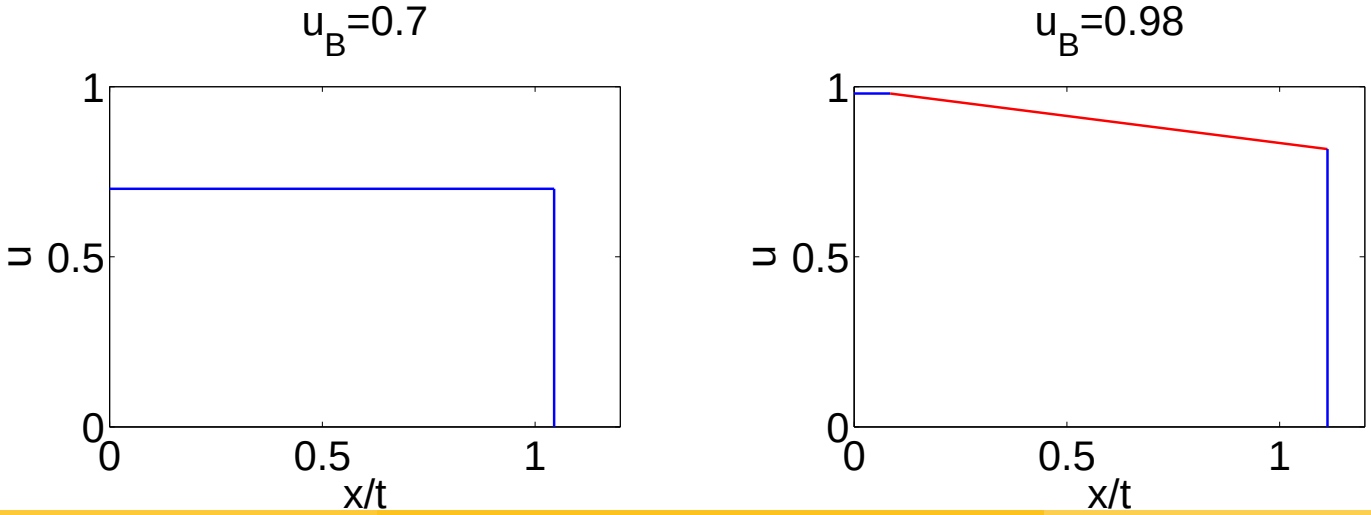
Figure: The flux function and its derivative. left: $f(u) = \frac{u^2}{u^2+M(1-u)^2}$; right: $f'(u) = \frac{2Mu(1-u)}{(u^2+M(1-u)^2)^2}$. $\alpha = \sqrt{\frac{M}{M+1}}$. $\alpha \approx 0.8165$ for $M = 2$.

Monotone Solution for BL Equation

Entropy solution for the Riemann initial problem has been well studied [5, 7]. Let α be the solution of $f'(u) = \frac{f(u)}{u}$, i.e.,

$$\alpha = \sqrt{\frac{M}{M+1}}. \quad \alpha \approx 0.8165 \text{ for } M = 2.$$

1. If $0 < u_B \leq \alpha$, the entropy solution has a single shock at $\frac{x}{t} = \frac{f(u_B)}{u_B}$.
2. If $\alpha < u_B < 1$, the entropy solution contains a rarefaction between u_B and α for $f'(u_B) < \frac{x}{t} < f'(\alpha)$ and a shock at $\frac{x}{t} = \frac{f(\alpha)}{\alpha}$.



Experiment Results

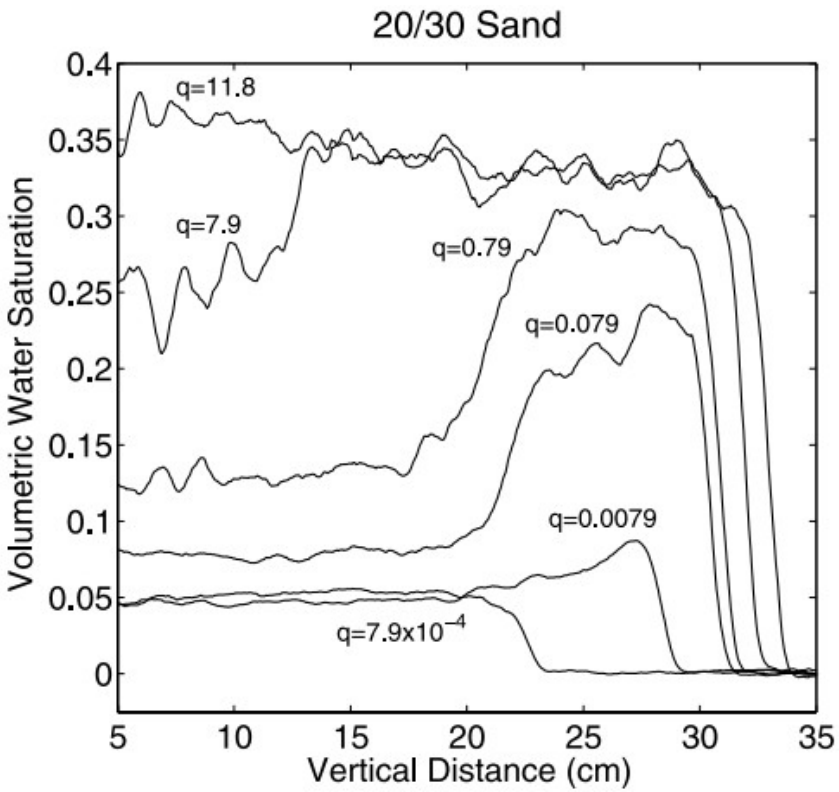


Figure: Snapshots of the saturation profile versus depth for six different applied fluxes in initially dry 20/30 sand measured using light transmission [2].

Overshoots ⇒ Nonmonotone profile ⇒ Modified BL Equation (MBL).

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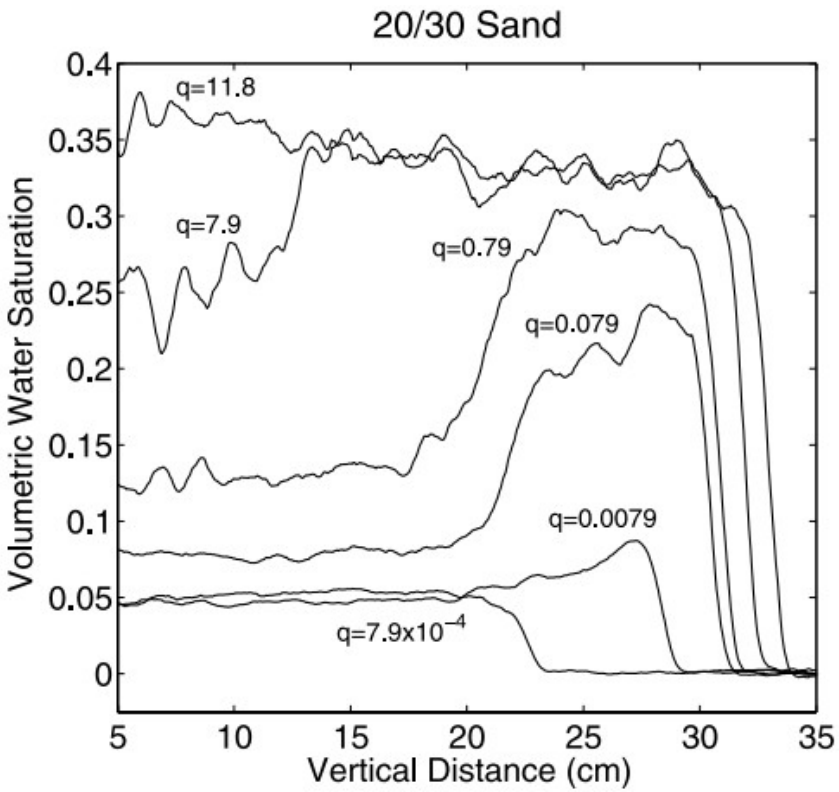

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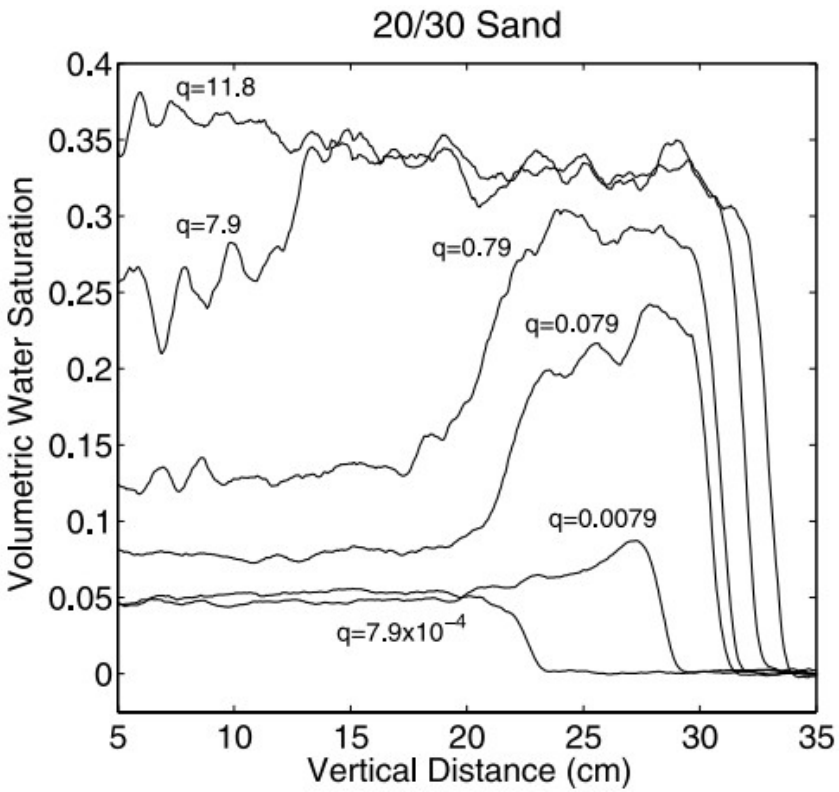

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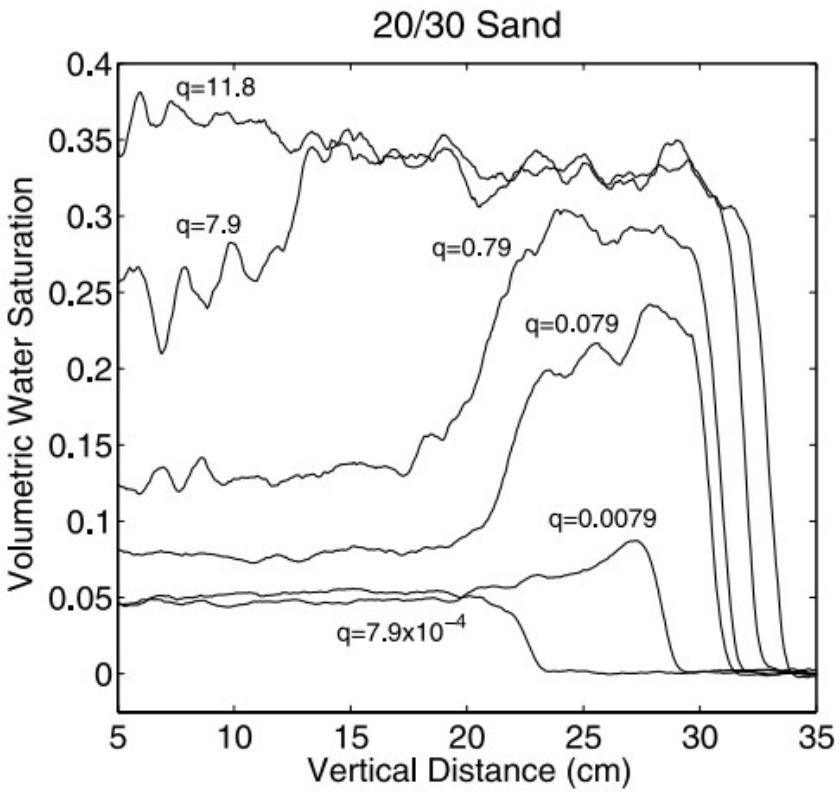

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Overshoots $\Rightarrow$ Nonmonotone profile $\Rightarrow$ Modified BL Equation (MBL).

Outlines

Modified BL Equation (MBL)

Hassanizadeh and Gray [3, 4] have included the extra terms to model the *dynamic* effects in the capillary pressure between the two phases, and 1-D MBL Equation reads as

$$u_t + f(u)_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt}, \quad \epsilon > 0,\ \tau > 0,$$

where

$$f(u) = \frac{u^2}{u^2 + M(1-u)^2}, \qquad M = \frac{\mu_w}{\mu_o}$$

- Classical second order viscous term ϵu_{xx}
- Third order mixed derivative term $\epsilon^2 \tau u_{xxt}$
- ϵ is the diffusion coefficient. (ϵ, τ) determine the type of the solution profile. When τ is larger than the threshold value τ_*, the solution profile is non-monotone.

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2-D MBL Equation

Two dimensional MBL Equation is

$$u_t + f(u)_x + g(u)_y = \epsilon \Delta u + \epsilon^2 \tau \Delta u_t$$

where

$$f(u) = \frac{u^2}{u^2 + M(1-u)^2},$$
$$g(u) = f(u)(1 - C(1-u)^2).$$

BL v.s. MBL

$$u_t + f(u)_x = 0$$

$$u_t + f(u)_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt}$$

- The classical BL equation is hyperbolic. The numerical schemes for hyperbolic equations have been well-developed.
- The MBL equation, however, is pseudo-parabolic.
  Van Duijn et al [6]: first order finite difference scheme.
  Wang et al [8, 7]: second- and third-order Godunov-type staggered central schemes (1-D MBL equation).

Difficulty

General convection-diffusion equation:

$$u_t + f(u)_x = \epsilon u_{xx}$$

In the convection dominated case, some numerical schemes have

- extensive numerical viscosity: the solution under-resolved
- spurious oscillations near the shock

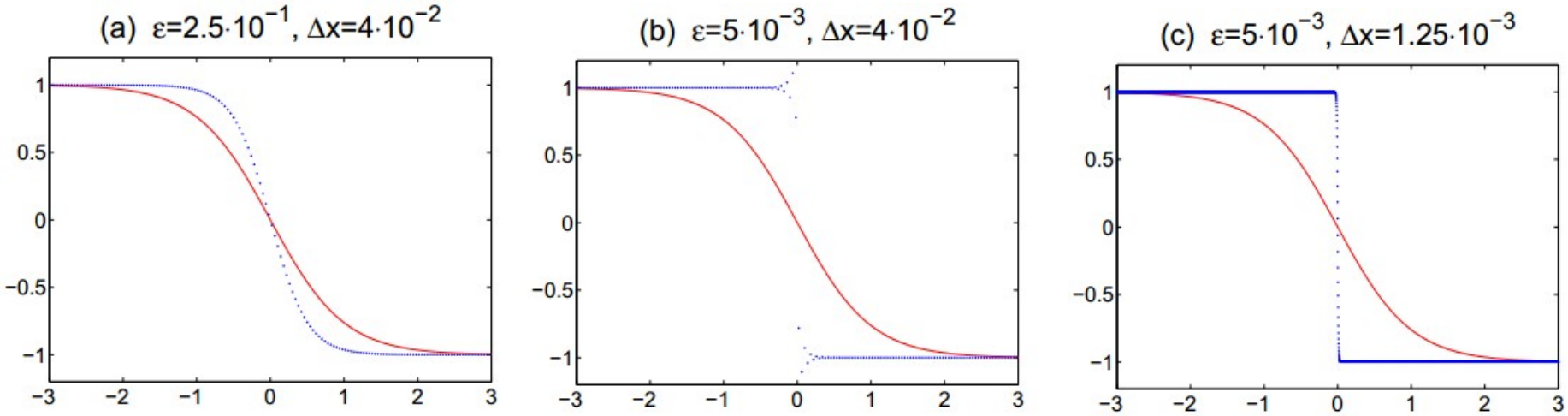

Figure: Solution of convection-diffusion equation. Solid line: the initial data.

To overcome these difficulties

- Fast Explicit Operator Splitting (FEOS) Method :
  - Chertock A, Kurganov A, Petrova G. Fast explicit operator splitting method for convection-diffusion equations. *International Journal for Numerical Methods in Fluids*, 2009, 59(3): 309-332.
  - Chertock A, Kurganov A. On splitting-based numerical methods for convection-diffusion equations. *Numerical Methods for Balance Laws, Aracne editrice Srl, Rome*, 2010.
- numerically preserving a delicate balance between the convection and diffusion terms.

Outlines

Fast Explicit Operator Splitting (FEOS) Method

$$u_t + f(u)_x = \epsilon u_{xx}$$

Split the equation into two sub-equations:

$$u_t + f(u)_x = 0$$
$$u_t = \epsilon u_{xx}$$

- Second-order Strang splitting method
- Nonlinear: hyperbolic problem $\rightarrow$ finite-volume Godunov-type scheme
- Linear:
  - exact solution of the heat equation $\rightarrow$ approximated by a conservative and accurate quadrature formula. *e.g.* midpoint rule
  - pseudo-spectral method
- Splitting timestep: $\Delta t \sim \Delta x$

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Splitting Strategy

Take 1-D case for simplicity, and the 2-D case can be done similarly.

How to split the equation?

$$u_t + f(u)_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt}$$

$$\begin{cases} u_t + f(u)_x = 0 \\ u_t \qquad\qquad = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt} \end{cases}$$

Not a good choice!
We want time derivative appear in both equations. Otherwise:

$$S \neq S_1 \circ S_2 + O((\Delta t)^2)$$

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Our Splitting Strategy

$$u_t + f(u)_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt}$$

Rewrite the MBL equation:

$$(u - \epsilon^2 \tau u_{xx})_t + f(u)_x = \epsilon u_{xx}.$$

S_N: The solution operator associated with the nonlinear hyperbolic equation with flux term

$$(u - \epsilon^2 \tau u_{xx})_t + f(u)_x = 0$$

S_L : associated with the linear equation with the diffusion term

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Splitting Strategy conti

Assume the solution of the MBL equation $u(x,t)$ is available at time t, then after a small time step Δt, a first order splitting method consists of two steps:

$$u(x,t+\Delta t) = S_L(\Delta t)S_N(\Delta t)u(x,t)$$

The second order operator splitting method consists of three steps:

$$u(x,t+\Delta t) = S_N(\frac{\Delta t}{2})S_L(\Delta t)S_N(\frac{\Delta t}{2})u(x,t)$$

S_N : Nonlinear Step

$$(u - \epsilon^2 \tau u_{xx})_t + f(u)_x = 0$$

$$\begin{cases} v_t + f(u)_x = 0, \\ u - \epsilon^2 \tau u_{xx} = v. \end{cases}$$

- Semi-discrete scheme:

$$\frac{dv_j(t)}{dt} = \frac{H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}}}{\Delta x}$$

$H_{j+\frac{1}{2}}$: Godunov-type central-upwind scheme. To discretize the flux:

  - nonlinear Minmod limiter
  - WENO5 reconstruction

- Integrate in time by third-order SSP Runge-Kutta method.

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S_N : Nonlinear Step conti.

$$\begin{cases} v_t + f(u)_x = 0, \\ u - \epsilon^2 \tau u_{xx} = v. \end{cases}$$

- At each stage of Runge-Kutta method, the elliptic equation is solved by the spectral method.
  1. Take Fast Fourier Transform (FFT)
  2. $$\tilde{u} - \epsilon^2 \tau (ik)^2 \tilde{u} = \tilde{v},$$
     $$(1 + \epsilon^2 \tau k^2)\tilde{u} = \tilde{v}.$$
     $$\tilde{u} = \frac{\tilde{v}}{1 + \epsilon^2 \tau k^2}$$
  3. Take the Inverse FFT

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S_L : Linear Step

$$(u - \epsilon^2 \tau u_{xx})_t = \epsilon u_{xx}$$

Linear equation, by using spectral method, we get

$$(\tilde{u} - \epsilon^2 \tau (ik)^2 \tilde{u})_t = \epsilon (ik)^2 \tilde{u},$$
$$\tilde{u}_t = \frac{-\epsilon k^2}{1 + \epsilon^2 \tau k^2} \tilde{u}.$$

Therefore,

$$\tilde{u}(x, t + \Delta t) = \exp\left(\frac{-\epsilon k^2 \Delta t}{1 + \epsilon^2 \tau k^2}\right) \tilde{u}(x, t)$$

Outlines

Linear Problem

Test the accuracy on solving the equation with linear flux

$$u_t + au_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt},$$

In 2D, we consider the equation

$$u_t + au_x + bu_y = \epsilon \Delta u + \epsilon^2 \tau (\Delta u)_t.$$

Initial condition is

$$u(x,0) = \sin(\pi x) \qquad x \in [0,2]$$

$$u(x,y,0) = \sin(\pi x) + \sin(\pi y) \qquad (x,y) \in [0,2] \times [0,2]$$

1D

$$u_t + au_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt}$$

The accuracy test for second-order central-upwind scheme for 1D equation with $a = 1$, $\epsilon = 10^{-3}$ and $\tau = 5$.

N	L_1 error	order	L_2 error	order	L_∞ error	order
64	1.4755E-02	-	1.3400E-02	-	2.4467E-02	-
128	2.6529E-03	2.4755	2.4454E-03	2.4541	5.9092E-03	2.0498
256	4.5606E-04	2.5403	3.7676E-04	2.6983	9.7694E-04	2.5966
512	1.0240E-04	2.1551	8.0050E-05	2.2347	1.1068E-04	3.1418
1024	2.5122E-05	2.0272	1.9691E-05	2.0233	1.9653E-05	2.4936
2048	6.2732E-06	2.0017	4.9248E-06	1.9994	4.9236E-06	1.9969

The convergence rate is second order.

1D

$$u_t + au_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt}$$

The accuracy test for WENO5 scheme for 1D equation with $a = 1$, $\epsilon = 10^{-3}$ and $\tau = 5$.

N	L_1 error	order	L_2 error	order	L_∞ error	order
64	1.3145E-05	-	1.0293E-05	-	1.0782E-05	-
128	8.6308E-07	3.9289	6.7674E-07	3.9269	6.7037E-07	4.0076
256	8.3592E-08	3.3681	6.5634E-08	3.3661	6.4986E-08	3.3667
512	9.6942E-09	3.1082	7.6128E-09	3.1079	7.5732E-09	3.1012
1024	1.1924E-09	3.0233	9.3638E-10	3.0233	9.3454E-10	3.0186
2048	1.5306E-10	2.9617	1.2021E-10	2.9616	1.2057E-10	2.9544

The convergence rate is third order.

2D

$$u_t + au_x + bu_y = \epsilon \Delta u + \epsilon^2 \tau (\Delta u)_t$$

The accuracy test for WENO5 scheme for 2D equation with $a = 1$, $b = 1$, $\epsilon = 10^{-3}$ and $\tau = 5$.

N	L_1 error	order	L_2 error	order	L_∞ error	order
64	3.3396E-05	-	2.0586E-05	-	2.1565E-05	-
128	2.1915E-06	3.9297	1.3535E-06	3.9269	1.3407E-06	4.0076
256	2.1273E-07	3.3648	1.3127E-07	3.3661	1.2997E-07	3.3667
512	2.4679E-08	3.1077	1.5226E-08	3.1079	1.5146E-08	3.1012
1024	3.0370E-09	3.0226	1.8736E-09	3.0226	1.8690E-09	3.0185

The convergence rate is third order.

Nonlinear Problem

We first solve the one-dimensional MBL equation

$$u_t + f(u)_x = \epsilon u_{xx} + \epsilon^2 \tau u_{xxt},$$

with the initial condition

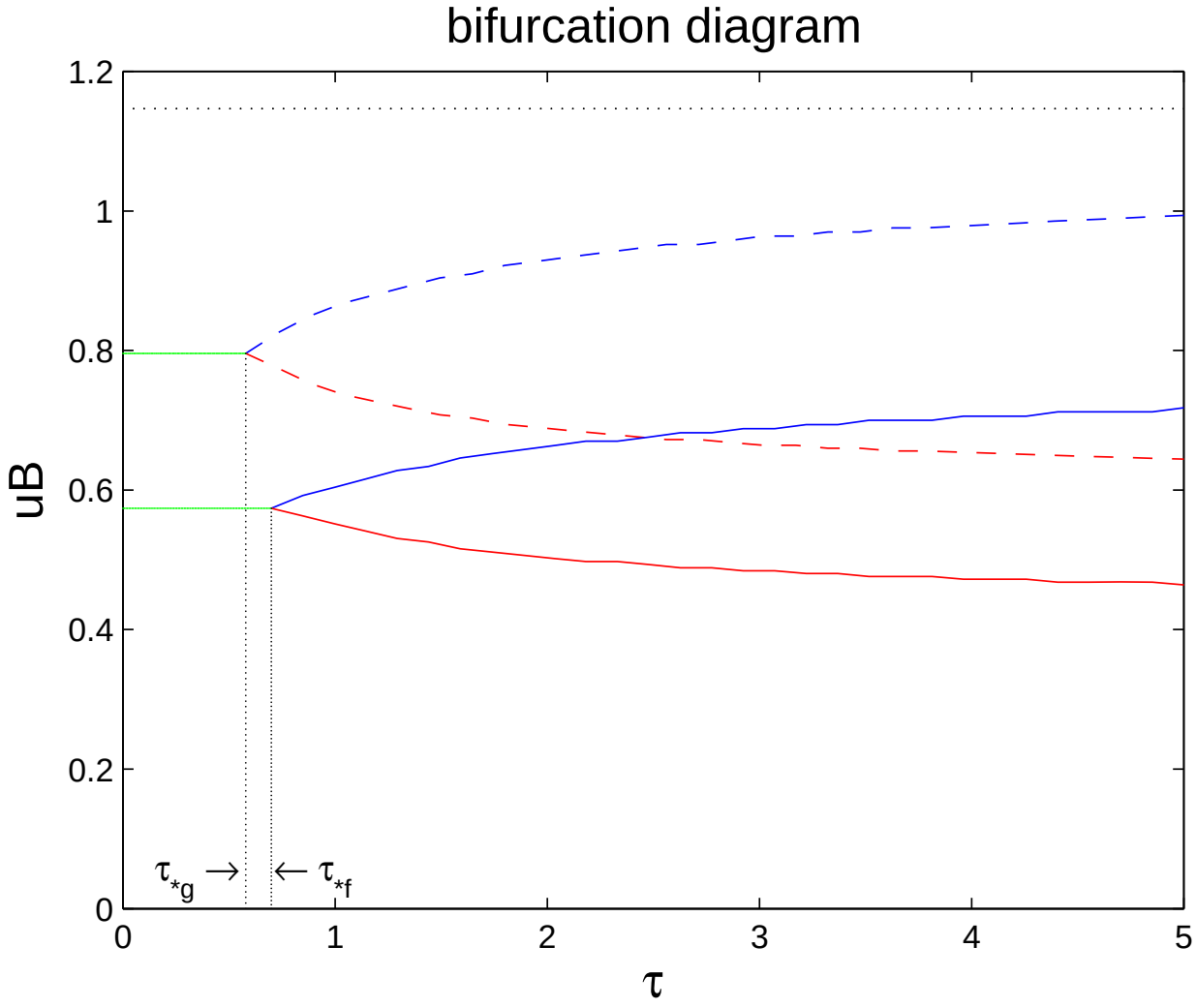
$$u(x,0) = \begin{cases} u_B, & \text{if } x \in (0.75, 2.25), \\ 0, & \text{otherwise}, \end{cases}$$

on the domain $[0,3]$ with periodic boundary condition.

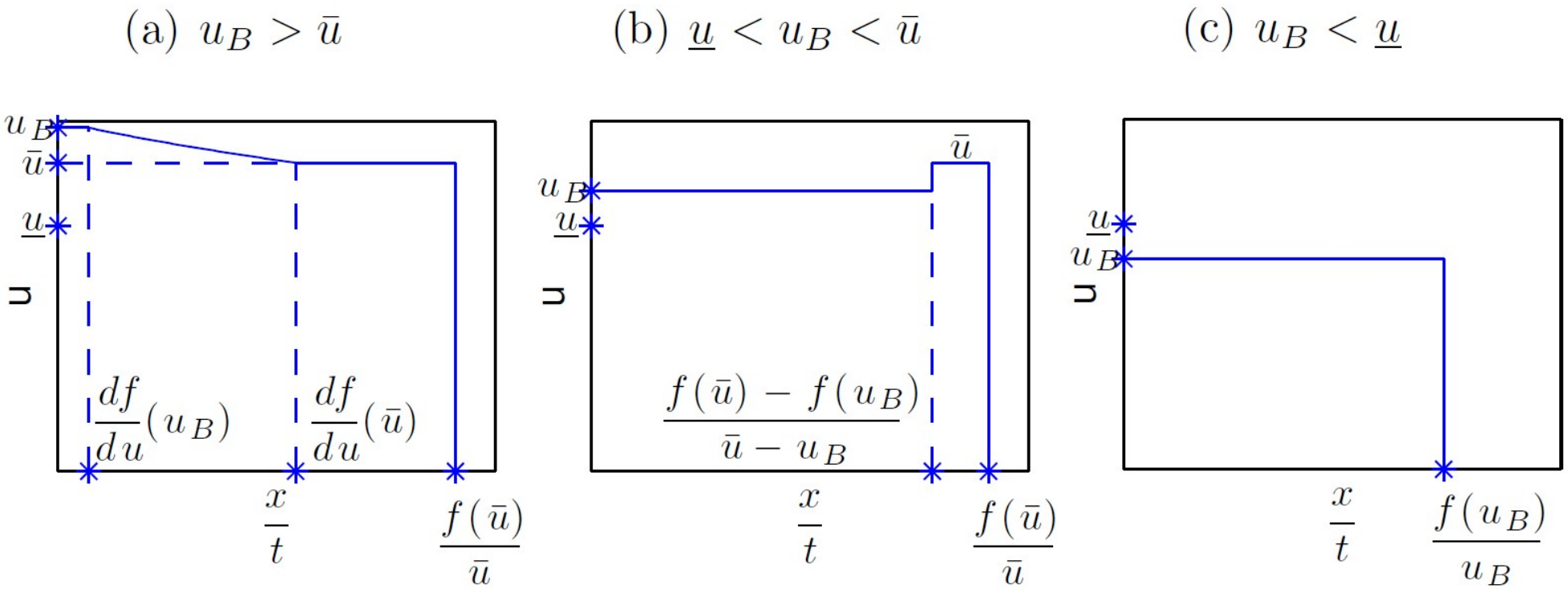
$$f(u) = \frac{u^2}{u^2 + M(1-u)^2}$$

Here, $\epsilon = 10^{-3}$, $M = 1/2$, and final time $T = 0.5$.

This problem has been studied in [6]. Van Duijn et al numerically provided a bifurcation diagram of MBL equation as τ and u_B vary.
$M = \frac{1}{2}$, $C = 2$. Blue: $\bar{u}$, Red: $\underline{u}$, Solid: f, Dash g.



Comparison between second order central-upwind and WENO5 methods

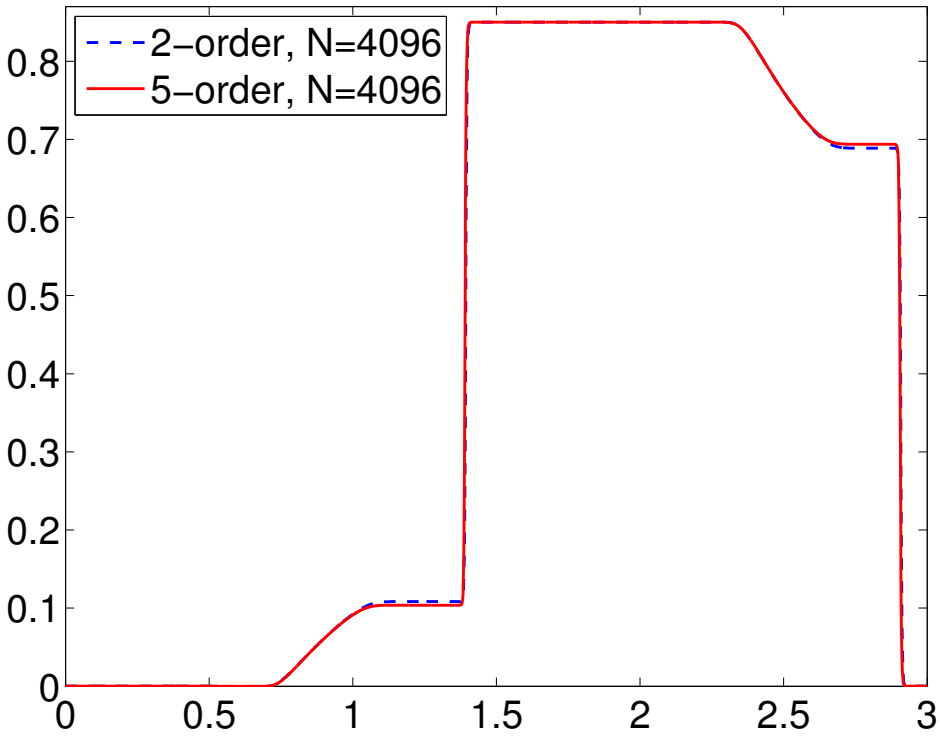


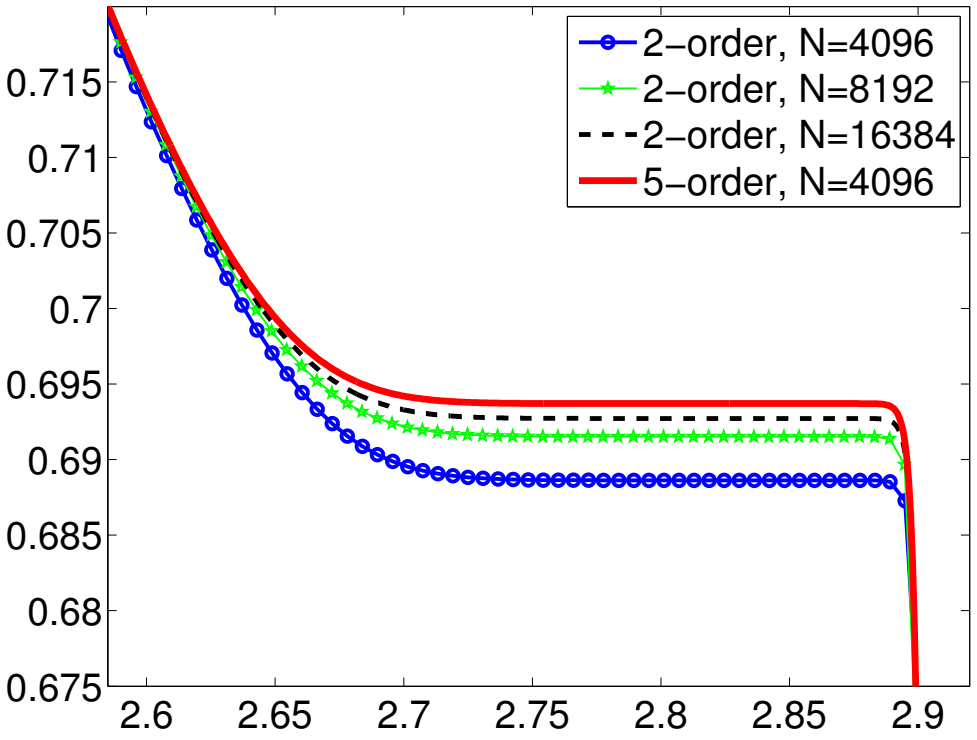
(a) $u_B > \bar{u} \Rightarrow$ rarefaction + shock

(b) $\underline{u} < u_B < \bar{u} \Rightarrow$ jump up + jump down (shock)
(Oscillation may appear near $u = u_B$)

(c) $u_B < \underline{u} \Rightarrow$ single shock
(Oscillation may appear near $u = u_B$)

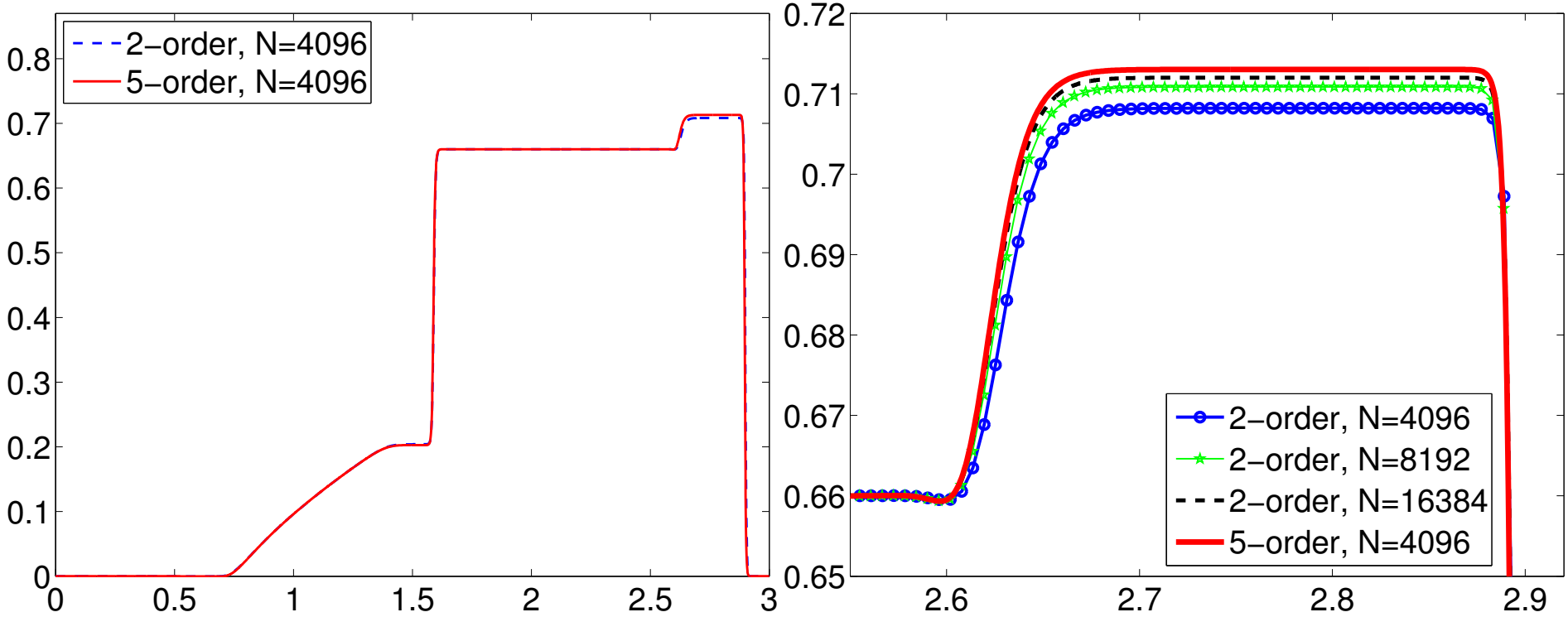
Case 1: $u_B > \bar{u}$: $u_B = 0.85, \tau = 3.5$





By bifurcation diagram: $\bar{u} \approx 0.698$

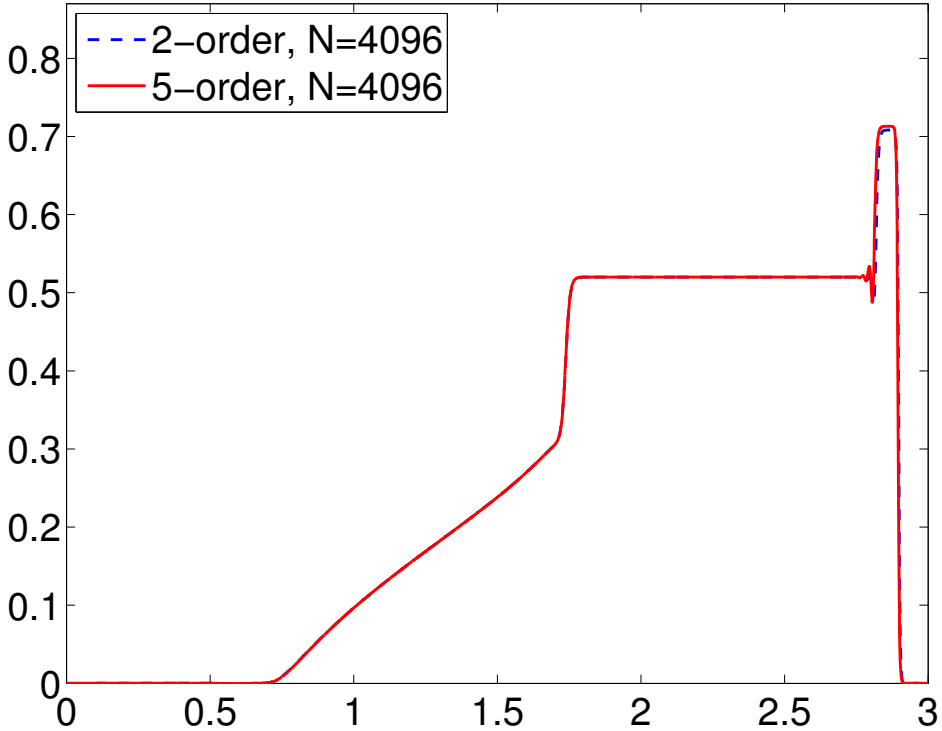
Case 2: $\underline{u} < u_B < \bar{u}$: $u_B = 0.66, \tau = 5$

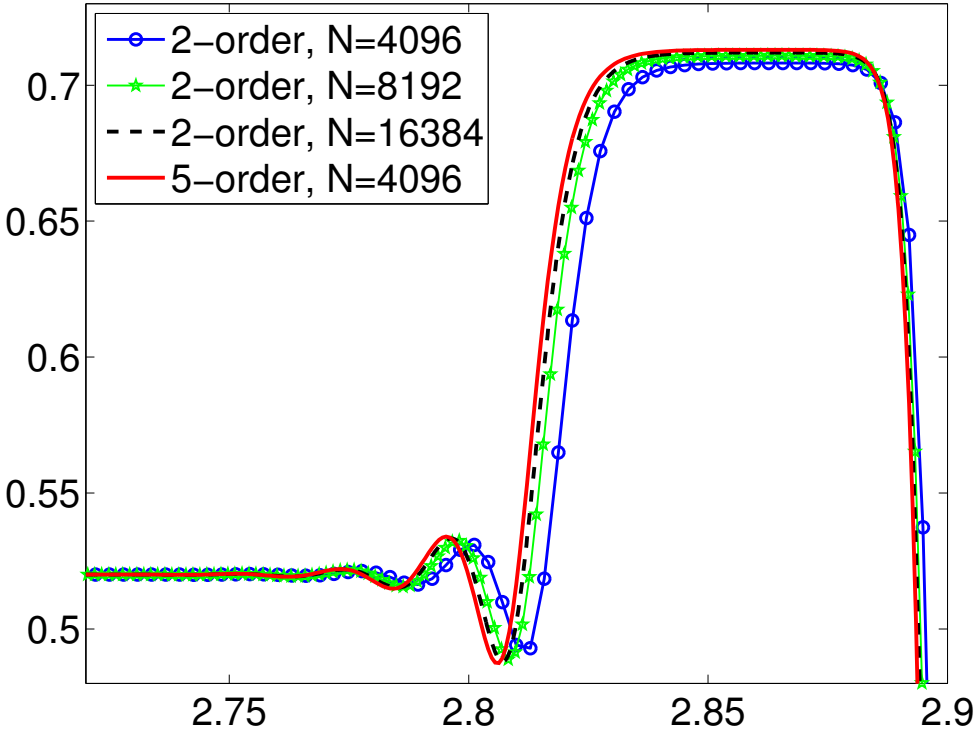


By bifurcation diagram: $\bar{u} \approx 0.713$

non-monoton solution profile.

Case 3: $\underline{u} < u_B < \bar{u}$: $u_B = 0.52, \tau = 5$





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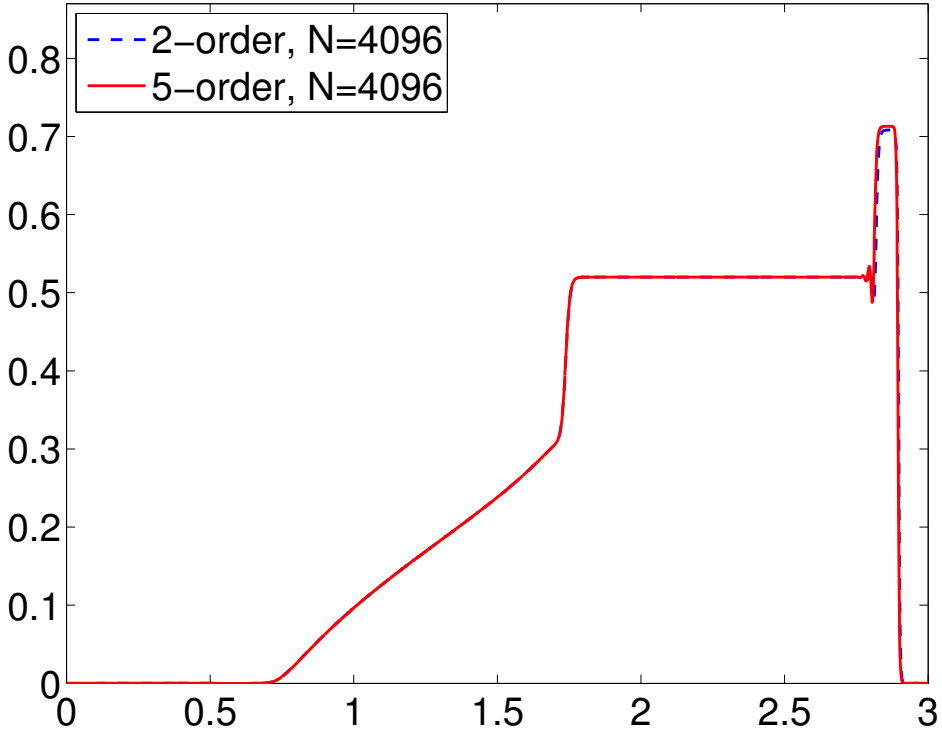
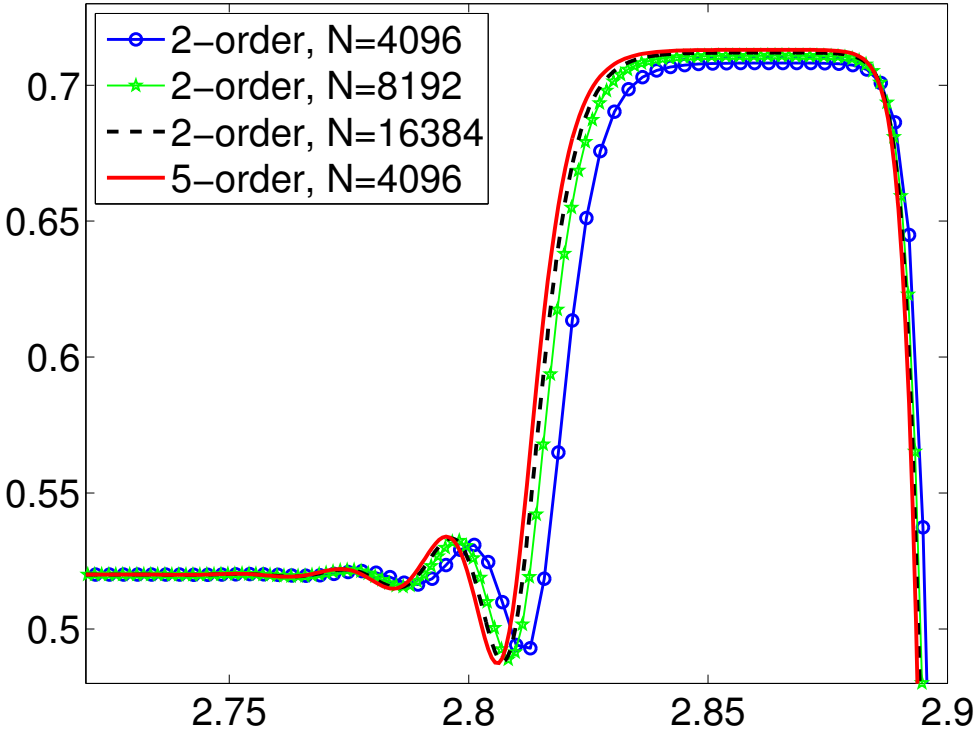
non-monoton solution profile. Oscillation?

Case 3: $\underline{u} < u_B < \bar{u}$: $u_B = 0.52, \tau = 5$





By bifurcation diagram: $\bar{u} \approx 0.713$

non-monoton solution profile. Oscillation?

Outlines

Flux $f(u)$ v.s. $g(u)$

$$u_t + f(u)_x + g(u)_y = \epsilon \Delta u + \epsilon^2 \tau \Delta u_t$$

where

$$f(u) = \frac{u^2}{u^2 + M(1-u)^2},$$
$$g(u) = f(u)(1 - C(1-u)^2).$$

In our computations, we take $C = 2$.

Computation domain is $[0, 13]$ and the initial value is

$$u_0(x) = \begin{cases} u_B & x \in [4, 10], \\ 0 & \text{otherwise.} \end{cases}$$

$T = 1.2, \quad N = 16384.$

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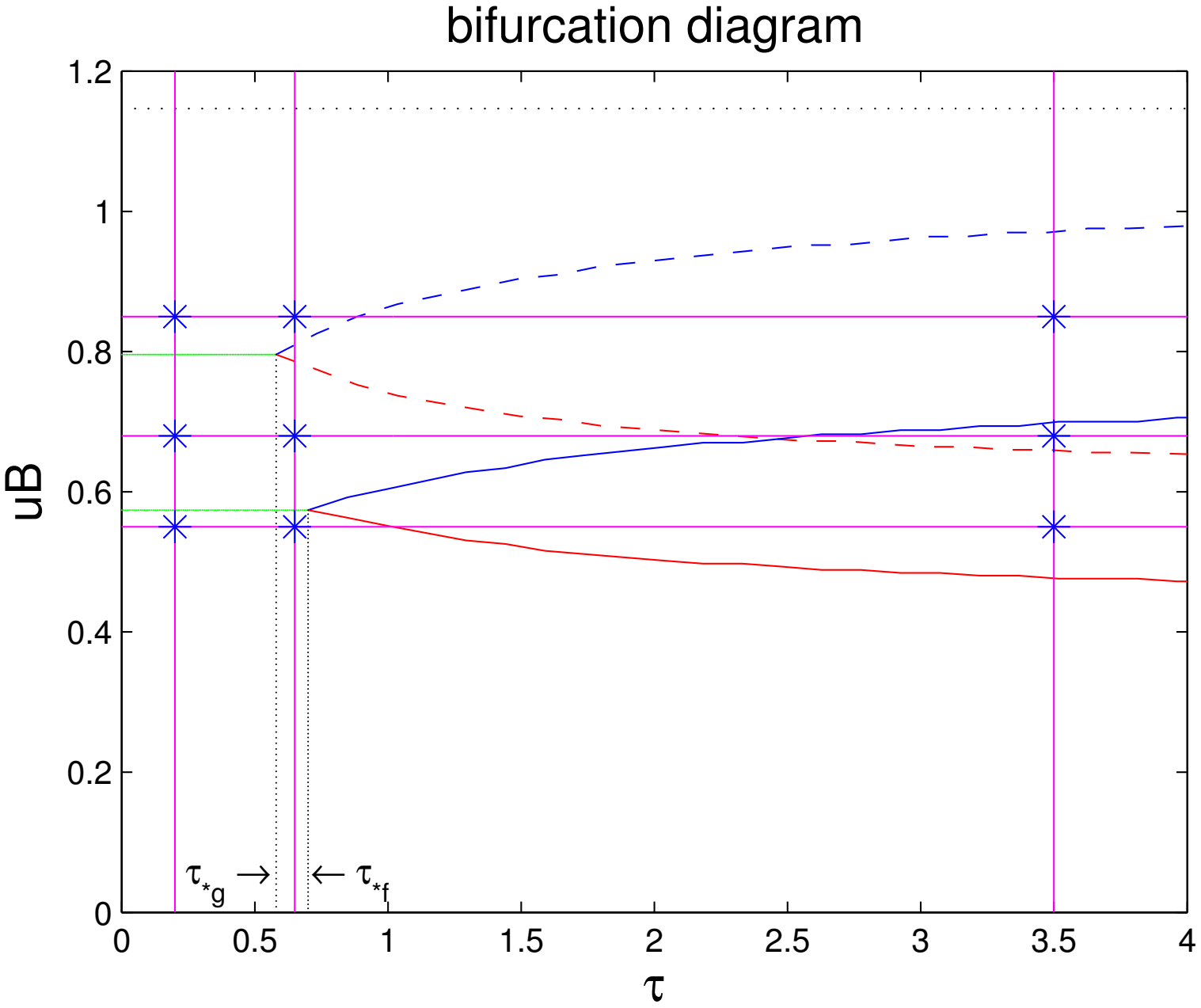
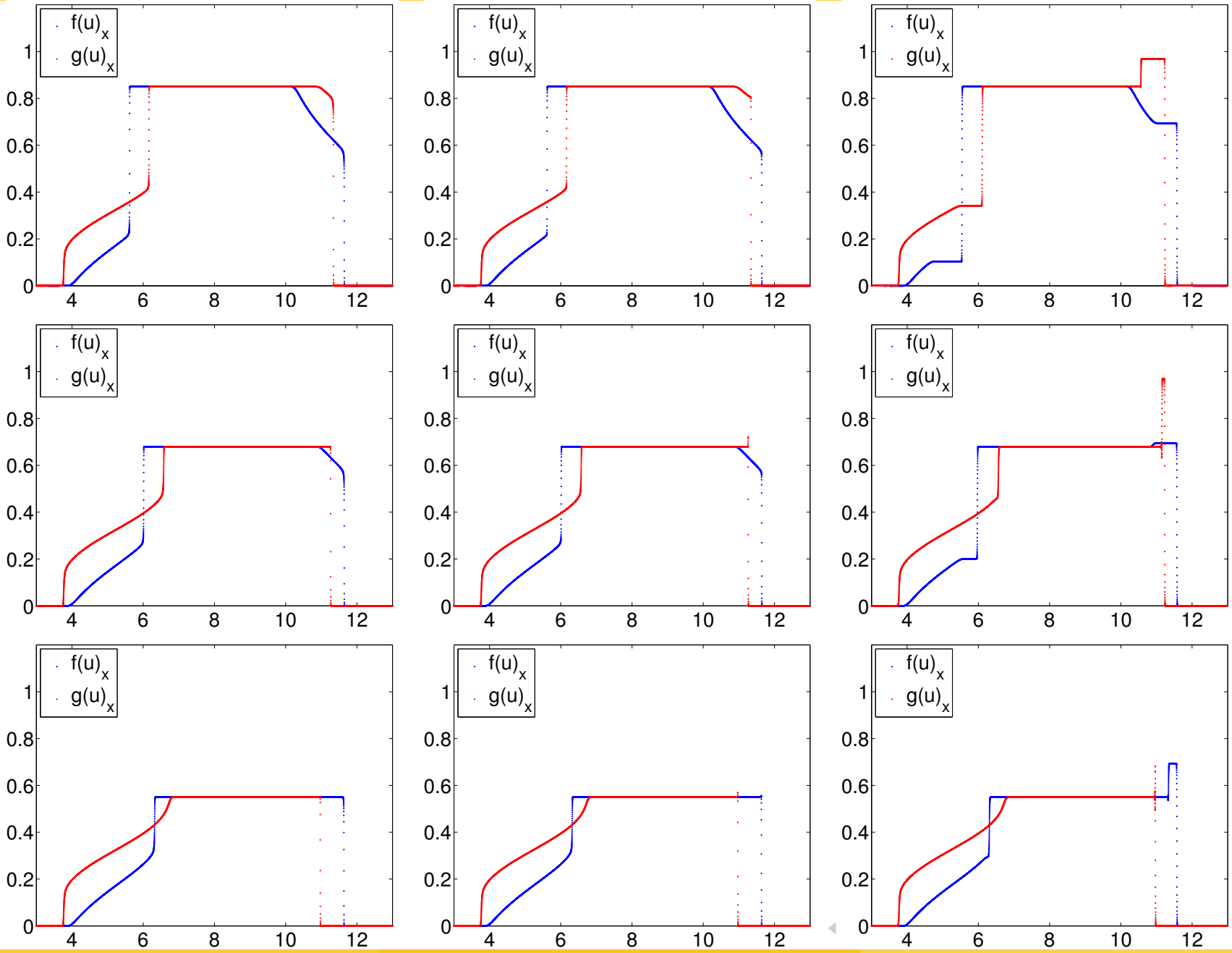
In order to compute the solution profiles associated with two different fluxes f and g, we choose nine representative pairs of (τ, u_B) values.

$(0.2, 0.85)$	$(0.65, 0.85)$	$(3.5, 0.85)$
$(0.2, 0.68)$	$(0.65, 0.68)$	$(3.5, 0.68)$
$(0.2, 0.55)$	$(0.65, 0.55)$	$(3.5, 0.55)$


bifurcation diagram
uB
τ
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
τ*g →
← τ*f

Comparison between flux functions



2-D Rotational BL and MBL Equations

$$u_t + \nabla \cdot \left(\vec{V} \frac{u^2}{u^2 + M(1-u)^2} \right) = h(\Delta u, \Delta u_t)$$

where $\vec{V}(x) = [y, -x]$, and $M = 2$ with the initial condition

$$u(x, y, 0) = \begin{cases} \sqrt{\frac{2}{3}}, & \text{if } x^2 + y^2 \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Our computational domain is $[-2, 2]^2$.

Classical BL Equation: $h(\Delta u, \Delta u_t) = 0$
Modified BL Equation: $h(\Delta u, \Delta u_t) = \epsilon \Delta u + \epsilon^2 \tau \Delta u_t$
Here, $\epsilon = 10^{-3}$ and $\tau = 5$.

2-D Examples

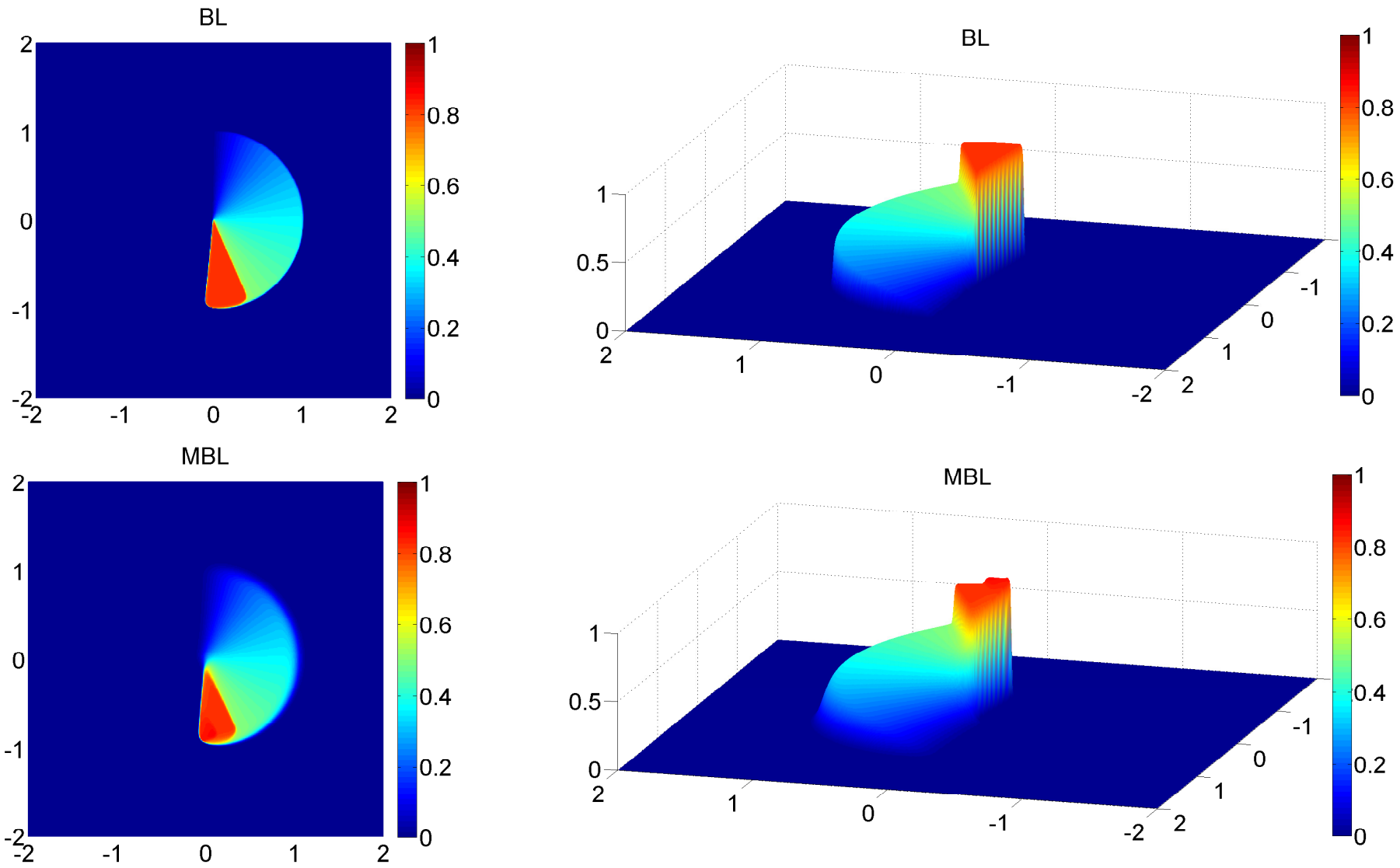


Figure: BL v.s. MBL equation: view from the top (left) and 3D view (right).

2-D BL and MBL Equations (1)

$$u_t + f(u)_x + g(u)_y = h(\Delta u, \Delta u_t)$$

where

$$f(u) = \frac{u^2}{u^2 + M(1-u)^2},$$
$$g(u) = f(u)(1 - 2(1-u)^2),$$

with two different initial conditions. The first initial condition is a smooth two-dimensional Gaussian function

$$u(x, y, 0) = 5e^{-20(x^2+y^2)}$$

cut off by a plateau $u = 0.85$ in the computational domain $[-1.25, 1.25]^2$ with $\tau = 2.5, M = 1/2, \epsilon = 10^{-3}$.

2-D Examples

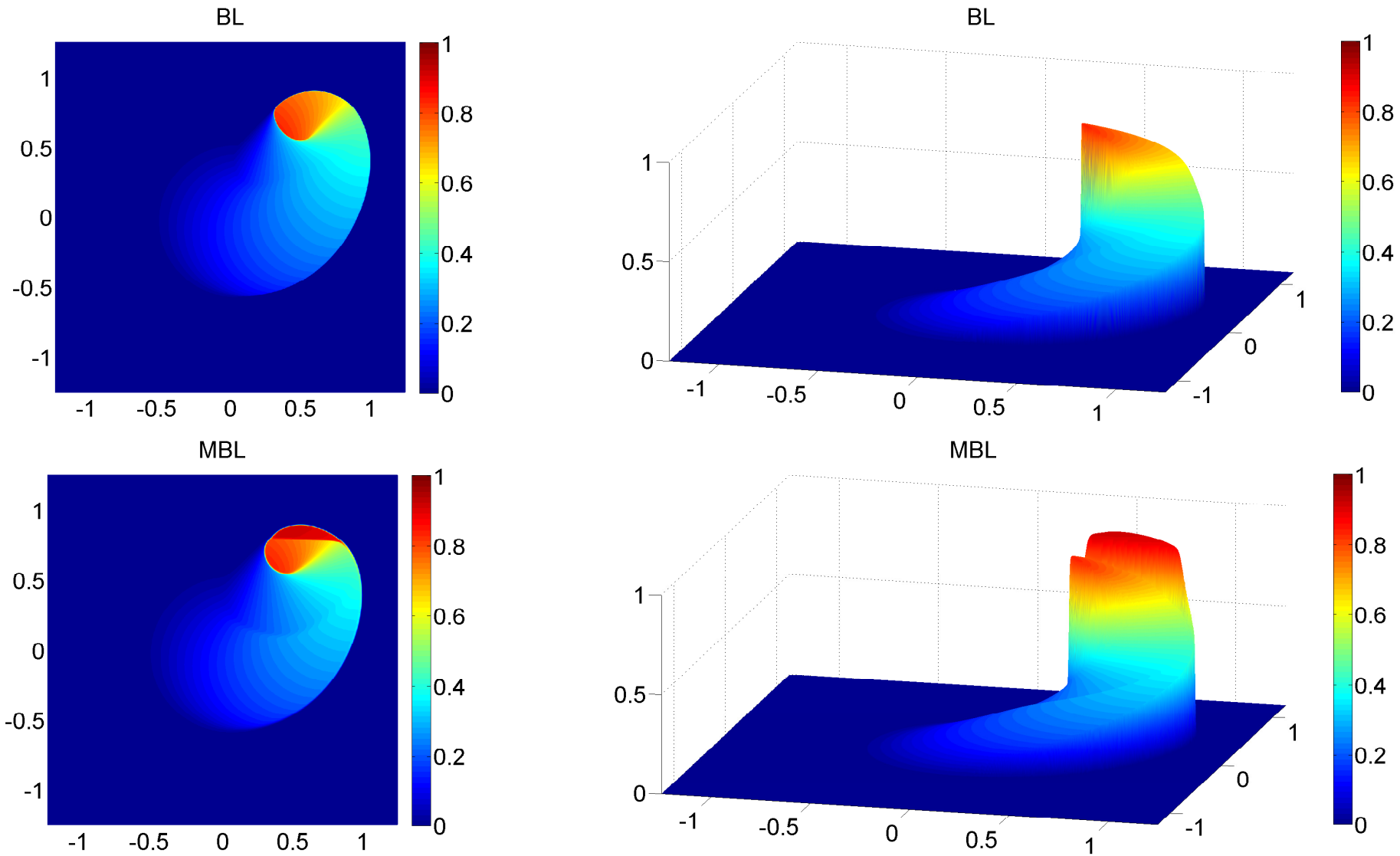


Figure: BL v.s. MBL equation: view from the top (left) and 3D view (right).

2-D BL and MBL Equations (2)

$$u_t + f(u)_x + g(u)_y = h(\Delta u, \Delta u_t)$$

where

$$f(u) = \frac{u^2}{u^2 + M(1-u)^2},$$
$$g(u) = f(u)(1 - 2(1-u)^2),$$

The second initial condition is a nonsmooth function

$$u(x,y,0) = \begin{cases} u_B, & \text{if } \ 0.75 \le |x| \le 2.25, \quad \text{or} \quad 0.75 \le |y| \le 2.25, \\ 0, & \text{otherwise} \end{cases}$$

in the computational domain $[0,3]^2$ with $\tau = 2.5, M = 1/2$, and $u_B = 0.85$.

2-D Examples

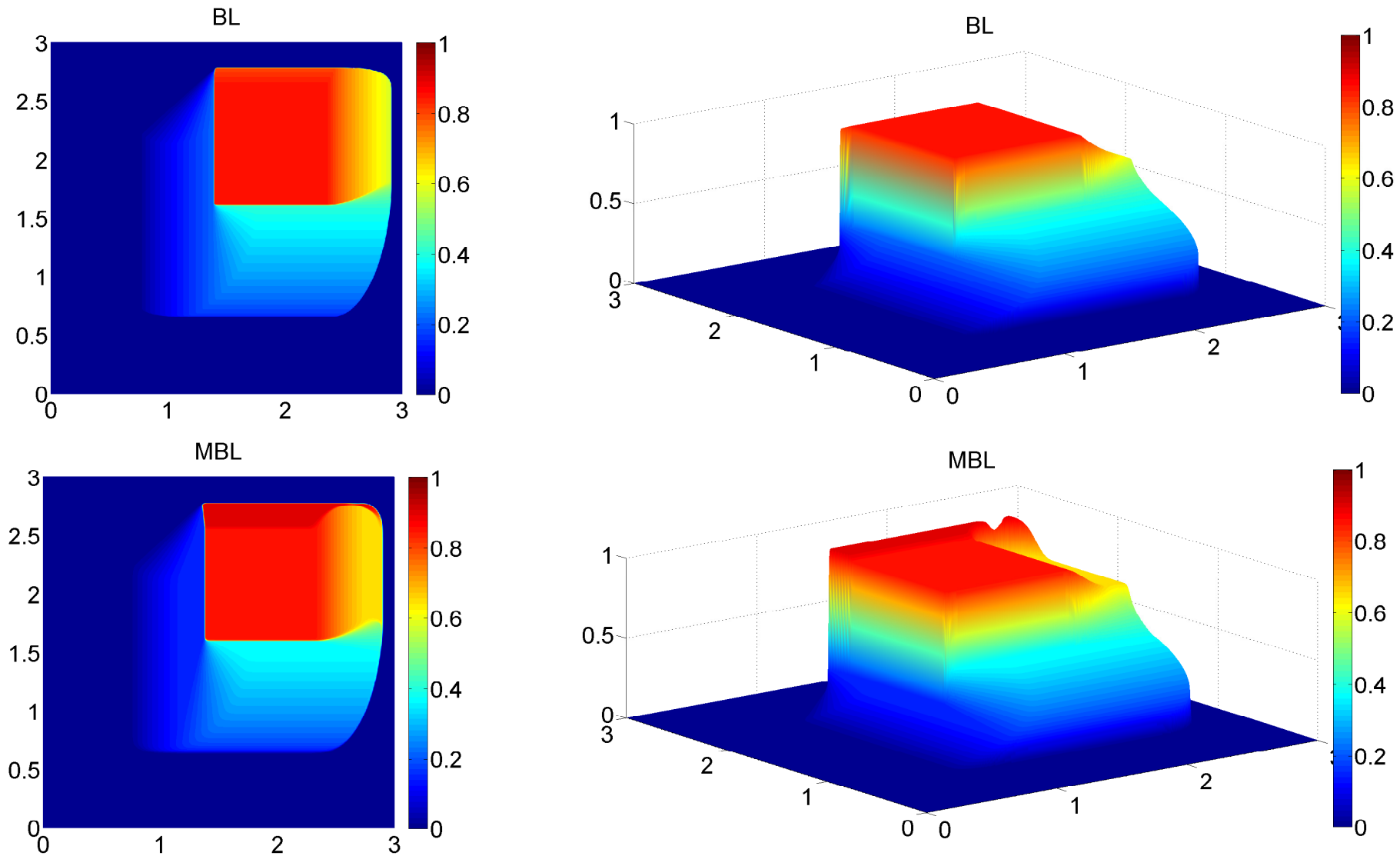


Figure: BL v.s. MBL equation: view from the top (left) and 3D view (right).

Outlines

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Thank you!